

# Moufang symmetry XI.

## Integrability of generalized Lie equations of continuous Moufang transformations

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### Abstract

Integrability of generalized Lie equations of continuous Moufang transformations is inquired.

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## 1 Introduction

In this paper we proceed explaining the Moufang symmetry. The present paper can be seen as a continuation of [1, 2, 3].

## 2 Generalized Lie equations

In [3] the *generalized Lie equations* (GLE) of continuous Moufang transformations were found. For  $S_g A$  the GLE read

$$u_j^s(g) \frac{\partial(S_g A)^\mu}{\partial g^s} + T_j^\nu(A) \frac{\partial(S_g A)^i}{\partial A^\nu} + P_j^\nu(S_g A) = 0 \quad (2.1a)$$

$$v_j^s(g) \frac{\partial(S_g A)^\mu}{\partial g^s} + P_j^\nu(h) \frac{\partial(S_g A)^i}{\partial A^\nu} + T_j^\nu(S_g A) = 0 \quad (2.1b)$$

$$w_j^s(g) \frac{\partial(S_g A)^\mu}{\partial g^s} + S_j^\nu(h) \frac{\partial(S_g A)^i}{\partial A^\nu} + S_j^\nu(S_g A) = 0 \quad (2.1c)$$

where  $gh$  is the product of  $g$  and  $h$ , and the auxiliary functions  $u_j^s$ ,  $v_j^s$ ,  $w_j^s$  and  $S_j^\mu$ ,  $T_j^\mu$ ,  $P_j^\mu(g)$  are related with the constraints

$$u_j^s(g) + v_j^s(g) + w_j^s(g) = 0 \quad (2.2)$$

$$S_j^\mu(A) + T_j^\mu(A) + P_j^\mu(A) = 0 \quad (2.3)$$

For  $T_g A$  the GLE read

$$v_j^s(g) \frac{\partial(T_g A)^\mu}{\partial g^s} + S_j^\nu(A) \frac{\partial(T_g A)^i}{\partial A^\nu} + P_j^\nu(T_g A) = 0 \quad (2.4a)$$

$$u_j^s(g) \frac{\partial(T_g A)^\mu}{\partial g^s} + P_j^\nu(h) \frac{\partial(T_g A)^i}{\partial A^\nu} + S_j^\nu(T_g A) = 0 \quad (2.4b)$$

$$w_j^s(g) \frac{\partial(T_g A)^\mu}{\partial g^s} + T_j^\nu(h) \frac{\partial(T_g A)^i}{\partial A^\nu} + T_j^\nu(T_g A) = 0 \quad (2.4c)$$

In this paper we inquire integrability of GLE (2.1a–c) and (2.4a–c).

### 3 Generalized Maurer-Cartan equations and Yamagutian I

Recall from [1] that for  $x$  in  $T_e(G)$  the infinitesimal translations of  $G$  are defined by

$$L_x \doteq x^j u_j^s(g) \frac{\partial}{\partial g^s}, \quad R_x \doteq x^j v_j^s(g) \frac{\partial}{\partial g^s}, \quad M_x \doteq x^j w_j^s(g) \frac{\partial}{\partial g^s} \quad \in T_g(G)$$

with constriant

$$L_x + R_x + M_x = 0$$

Following triality [2] define the Yamagutian  $Y(x; y)$  by

$$6Y(x; y) = [L_x, L_y] + [R_x, R_y] + [M_x, M_y]$$

We know from [2] the generalized Maurer-Cartan equations:

$$[L_x, L_y] = L_{[x, y]} - 2[L_x, R_y] \tag{3.1a}$$

$$[R_x, R_y] = R_{[y, x]} - 2[R_x, L_y] \tag{3.1b}$$

$$[L_x, R_y] = [R_x, L_y], \quad \forall x, y \in T_e(G) \tag{3.1c}$$

The latter can be written [2] as follows:

$$[L_x, L_y] = 2Y(x; y) + \frac{1}{3}L_{[x, y]} + \frac{2}{3}R_{[x, y]} \tag{3.2a}$$

$$[L_x, R_y] = -Y(x; y) + \frac{1}{3}L_{[x, y]} - \frac{1}{3}R_{[x, y]} \tag{3.2b}$$

$$[R_x, R_y] = 2Y(x; y) - \frac{2}{3}L_{[x, y]} - \frac{1}{3}R_{[x, y]} \tag{3.2c}$$

Define the (secondary) auxiliary functions of  $G$  by

$$u_{jk}^s(g) \doteq u_k^p(g) \frac{\partial u_j^s(g)}{\partial g^p} - u_j^p(g) \frac{\partial u_k^s(g)}{\partial g^p}$$

$$v_{jk}^s(g) \doteq v_k^p(g) \frac{\partial v_j^s(g)}{\partial g^p} - v_j^p(g) \frac{\partial v_k^s(g)}{\partial g^p}$$

$$w_{jk}^s(g) \doteq w_k^p(g) \frac{\partial w_j^s(g)}{\partial g^p} - w_j^p(g) \frac{\partial w_k^s(g)}{\partial g^p}$$

The Yamaguti functions  $Y_{jk}^i$  are defined by

$$6Y_{jk}^s(g) \doteq u_{jk}^s(g) + v_{jk}^s(g) + w_{jk}^s(g)$$

Evidently,

$$[L_x, L_y] = -x^j y^k u_{jk}^s(g) \frac{\partial}{\partial g^s}$$

$$[R_x, R_y] = -x^j y^k v_{jk}^s(g) \frac{\partial}{\partial g^s}$$

$$[M_x, M_y] = -x^j y^k w_{jk}^s(g) \frac{\partial}{\partial g^s}$$

By adding the above formulae, we get

$$Y(x; y) = -x^j y^k Y_{jk}^s(g) \frac{\partial}{\partial g^s}$$

**Lemma 3.1.** *One has*

$$u_{jk}^i \doteq 2Y_{jk}^i + \frac{1}{3}C_{jk}^s(u_s^i + 2v_s^i) \quad (3.3a)$$

$$v_{jk}^i \doteq 2Y_{jk}^i - \frac{1}{3}C_{jk}^s(2u_s^i + v_s^i) \quad (3.3b)$$

$$w_{jk}^i \doteq 2Y_{jk}^i + \frac{1}{3}C_{jk}^s(u_s^i - v_s^i) \quad (3.3c)$$

*Proof.* To see (3.3a,b) use (3.2a,c) . To see (3.3c) calculate by using (3.2):

$$\begin{aligned} [M_x, M_y] &= [L_x + R_x, L_y + R_y] \\ &= [L_x, L_y] + [L_x, R_y] + [R_x, L_y] + [R_x, R_y] \\ &= [L_x, L_y] + 2[L_x, R_y] + [R_x, R_y] \\ &= 2Y(x; y) + \frac{1}{3}(L_{[x,y]} - R_{[x,y]}) \end{aligned}$$

and (3.3b) easily follows.  $\square$

## 4 Generalized Maurer-Cartan equations and Yamagutian II

Recall from [3] that for  $x$  in  $T_e(G)$  the infinitesimal translations of  $G$  are defined by

$$S_x \doteq x^j S_j^\nu(A) \frac{\partial}{\partial A^\nu}, \quad T_x \doteq x^j T_j^\nu(A) \frac{\partial}{\partial A^\nu}, \quad P_x \doteq x^j P_j^\nu(A) \frac{\partial}{\partial A^\nu} \in T_A(\mathfrak{X})$$

with constriant

$$S_x + T_x + P_x = 0$$

Following triality [2] define the Yamagutian  $Y(x; y)$  by

$$6Y(x; y) = [S_x, S_y] + [T_x, T_y] + [P_x, P_y]$$

We know from [3] the generalized Maurer-Cartan equations:

$$[S_x, S_y] = S_{[x,y]} - 2[S_x, T_y] \quad (4.1a)$$

$$[T_x, T_y] = T_{[y,x]} - 2[T_x, S_y] \quad (4.1b)$$

$$[S_x, T_y] = [T_x, S_y], \quad \forall x, y \in T_e(G) \quad (4.1c)$$

The latter can be written [2] as follows:

$$[S_x, S_y] = 2Y(x; y) + \frac{1}{3}S_{[x,y]} + \frac{2}{3}T_{[x,y]} \quad (4.2a)$$

$$[T_x, T_y] = -Y(x; y) + \frac{1}{3}S_{[x,y]} - \frac{1}{3}T_{[x,y]} \quad (4.2b)$$

$$[T_x, T_y] = 2Y(x; y) - \frac{2}{3}S_{[x,y]} - \frac{1}{3}T_{[x,y]} \quad (4.2c)$$

Define the (secondary) auxiliary functions of  $G$  by

$$\begin{aligned} S_{jk}^\mu(A) &\doteq S_k^\nu(A) \frac{\partial S_j^\mu(A)}{\partial A^\nu} - S_j^\nu(g) \frac{\partial S_k^\mu(A)}{\partial A^\nu} \\ T_{jk}^\mu(A) &\doteq T_k^\nu(A) \frac{\partial T_j^\mu(A)}{\partial A^\nu} - T_j^\nu(g) \frac{\partial T_k^\mu(A)}{\partial A^\nu} \\ P_{jk}^\mu(A) &\doteq P_k^\nu(A) \frac{\partial P_j^\mu(A)}{\partial A^\nu} - P_j^\nu(g) \frac{\partial P_k^\mu(A)}{\partial A^\nu} \end{aligned}$$

The Yamaguti functions  $Y_{jk}^\mu$  are defined by

$$6Y_{jk}^\mu(A) \doteq S_{jk}^\mu(A) + v_{jk}^\mu(A) + P_{jk}^s(A)$$

Evidently,

$$\begin{aligned} [S_x, S_y] &= -x^j y^k S_{jk}^\nu(g) \frac{\partial}{\partial A^\nu} \\ [T_x, T_y] &= -x^j y^k T_{jk}^\nu(g) \frac{\partial}{\partial A^\nu} \\ [P_x, P_y] &= -x^j y^k P_{jk}^\nu(g) \frac{\partial}{\partial A^\nu} \end{aligned}$$

By adding the above formulae, we get

$$Y(x; y) = -x^j y^k Y_{jk}^\nu(A) \frac{\partial}{\partial A^\nu}$$

**Lemma 4.1.** *One has*

$$S_{jk}^\mu \doteq 2Y_{jk}^\mu + \frac{1}{3}C_{jk}^s(S_s^\mu + 2T_s^\mu) \quad (4.3a)$$

$$T_{jk}^\mu \doteq 2Y_{jk}^\mu - \frac{1}{3}C_{jk}^s(2S_s^\mu + T_s^\mu) \quad (4.3b)$$

$$P_{jk}^\mu \doteq 2Y_{jk}^\mu + \frac{1}{3}C_{jk}^s(S_s^\mu - T_s^\mu) \quad (4.3c)$$

*Proof.* To see (3.3a,b) use (3.2a,c) . To see (3.3c) calculate by using (3.2):

$$\begin{aligned} [P_x, P_y] &= [S_x + T_x, S_y + T_y] \\ &= [S_x, S_y] + [S_x, T_y] + [T_x, S_y] + [T_x, T_y] \\ &= [S_x, S_y] + 2[S_x, T_y] + [T_x, T_y] \\ &= 2Y(x; y) + \frac{1}{3} (S_{[x,y]} - T_{[x,y]}) \end{aligned}$$

and (4.3c) easily follows. □

## 5 Integrability conditions

**Theorem 5.1.** *The integrability conditons of the GLE (2.1a-c) (2.4a-c) read, respectively,*

$$Y_{jk}^s(g) \frac{\partial(S_g A)^\mu}{\partial g^s} + Y_{jk}^\nu(A) \frac{\partial(S_g A)^\mu}{\partial A^\nu} = Y_{jk}^\mu(S_g A) \quad (5.1a)$$

$$Y_{jk}^s(g) \frac{\partial(T_g A)^\mu}{\partial g^s} + Y_{jk}^\nu(A) \frac{\partial(T_g A)^\mu}{\partial A^\nu} = Y_{jk}^\mu(T_g A) \quad (5.1b)$$

*Proof.* We differentiate the GLE and use

$$\frac{\partial^2(S_g A)^\mu}{\partial g^j \partial g^k} = \frac{\partial^2(S_g A)^\mu}{\partial g^k \partial g^j}, \quad \frac{\partial^2(S_s A)^\mu}{\partial g^j \partial A^\nu} = \frac{\partial^2(S_g A)^\mu}{\partial A^\nu \partial g^j}, \quad \frac{\partial^2(S_g A)^\mu}{\partial A^\lambda \partial A^\nu} = \frac{\partial^2(S_g A)^\mu}{\partial A^\nu \partial A^\lambda} \quad (5.2a)$$

$$\frac{\partial^2(T_g A)^\mu}{\partial g^j \partial g^k} = \frac{\partial^2(T_g A)^\mu}{\partial g^k \partial g^j}, \quad \frac{\partial^2(T_s A)^\mu}{\partial g^j \partial A^\nu} = \frac{\partial^2(T_g A)^\mu}{\partial A^\nu \partial g^j}, \quad \frac{\partial^2(T_g A)^\mu}{\partial A^\lambda \partial A^\nu} = \frac{\partial^2(T_g A)^\mu}{\partial A^\nu \partial A^\lambda} \quad (5.2b)$$

First differentiate (2.1a) with respect to  $g^p$  and  $A^\lambda$ :

$$\frac{\partial v_j^s(g)}{\partial g^p} \frac{\partial (S_g A)^\mu}{\partial g^s} + v_j^s(g) \frac{\partial^2 (S_g A)^\mu}{\partial g^p \partial g^s} + P_j^\nu(A) \frac{\partial^2 (S_g A)^\mu}{\partial g^p \partial A^\nu} + \frac{\partial T_j^\mu(S_g A)}{\partial (S_g A)^\nu} \frac{\partial (S_g A h)^\nu}{\partial g^p} = 0 \quad (5.3a)$$

$$v_j^s(g) \frac{\partial^2 (S_g A)^\mu}{\partial A^\lambda \partial g^s} + \frac{\partial P_j^\nu(A)}{\partial A^\lambda} \frac{\partial (S_g A)^\mu}{\partial A^\nu} + P_j^\nu(A) \frac{\partial^2 (S_g A)^\mu}{\partial A h^\lambda \partial A^\nu} + \frac{\partial T_j^\mu(S_g A)}{\partial (S_g A)^\nu} \frac{\partial (S_g A)^\nu}{\partial A^\lambda} = 0 \quad (5.3b)$$

Now multiply (5.3a) by  $w_k^p(g)$  and (5.3b) by  $u_k^p(g)$  and add the resulting formulae. On the right hand side of the resulting formula use again the GLE (2.1a); then transpose the indexes  $j$  and  $k$  and subtract the result from the previous one. Then it turns out that due to (5.2a) all terms with the second order partial derivatives vanish and result reads

$$v_{jk}^s(g) \frac{\partial (S_g A)^\mu}{\partial g^s} + P_{jk}^\nu(A) \frac{\partial (S_g A)^\mu}{\partial A^\nu} = T_{jk}^\mu(S_g A) \quad (5.4)$$

By acting analogously with GLE (2.1b,c) we get

$$u_{jk}^s(g) \frac{\partial (S_g A)^\mu}{\partial g^s} + T_{jk}^\nu(A) \frac{\partial (S_g A)^\mu}{\partial A^\nu} = P_{jk}^\mu(S_g A) \quad (5.5a)$$

$$w_{jk}^s(g) \frac{\partial (S_g A)^\mu}{\partial g^s} + S_{jk}^\nu(A) \frac{\partial (S_g A)^\mu}{\partial A^\nu} = S_{jk}^\mu(S_g A) \quad (5.5b)$$

Now add (5.4), (5.5a) and (5.5b) to obtain (5.1a).

It remains to show that (5.4), (5.5a) and (5.5b) are equivalent to (5.1a). By using (4.3a–c) calculate

$$\begin{aligned} & v_{jk}^s(g) \frac{\partial (S_g A)^\mu}{\partial g^s} + P_{jk}^\nu(A) \frac{\partial (S_g A)^\mu}{\partial A^\nu} - T_{jk}^\mu(S_g A) \stackrel{(3.3b), (4.3b,c)}{=} \\ & u_{jk}^s(g) \frac{\partial (S_g A)^\mu}{\partial g^s} + T_{jk}^\nu(A) \frac{\partial (S_g A)^\mu}{\partial A^\nu} - P_{jk}^\mu(S_g A) \stackrel{(3.3a), (4.3b,c)}{=} \\ & w_{jk}^s(g) \frac{\partial (S_g A)^\mu}{\partial g^s} + S_{jk}^\nu(A) \frac{\partial (S_g A)^\mu}{\partial A^\nu} - S_{jk}^\mu(S_g A) \stackrel{(3.3c), (4.3a)}{=} \\ & = 2 \left( Y_{jk}^s(g) \frac{\partial (S_g A)^\mu}{\partial g^s} + Y_{jk}^\nu(A) \frac{\partial (S_g A)^\mu}{\partial A^\nu} - Y_{jk}^\mu(S_g A) \right) \end{aligned}$$

Integrability conditions (5.1b) are proved analogously by using (5.2b).  $\square$

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## References

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